PHYS5150 — PLASMA PHYSICS

LECTURE 21 - LANGMUIR WAVES

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1 LANGMUIR WAVES

In the last lecture we derived the expression for the plasma frequency by using the linearized continuity, momentum, and Poisson equations for the electron fluid:

$$\frac{\partial}{\partial t}\delta n = -n_e \frac{\partial}{\partial x}\delta v_e$$

$$\frac{\partial}{\partial t}\delta v_e = -\frac{e}{m_e}\delta E$$

$$\frac{\partial}{\partial x}\delta E = -\frac{e}{\epsilon_0}\delta n.$$

The resulting oscillations do not provide a full description of the physics going on under such conditions because the electrons have non-zero velocities and react differently on their spatial displacement. To account for this we need to the pressure gradient in the momentum equation, i.e.

$$\frac{\partial}{\partial t}\delta n = -n_e \frac{\partial}{\partial x}\delta v_e$$

$$\frac{\partial}{\partial t}\delta v_e = -\frac{e}{m_e}\delta E - \nabla p$$

$$\frac{\partial}{\partial x}\delta E = -\frac{e}{\epsilon_0}\delta n.$$

To relate ∇p to δn we need the equation of state. We assume that the electron temperature remains constant and that the pressure changes adiabatically which gives

$$\nabla p = \gamma k T_e \frac{\partial}{\partial x}\delta n,$$

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and, after substituting into the momentum equation

$$\frac{\partial}{\partial t}\delta v_e = -\frac{e}{m_e}\delta E - \gamma k T_e \frac{\partial}{\partial x}\delta n.$$

As before we eliminate δE and δv and find that

$$0 = \frac{\partial^2}{\partial t^2}\delta n - \frac{\gamma k T_e}{m_e}\frac{\partial^2}{\partial x^2}\delta n + \omega_{pe}^2 \delta n.$$

This time we obtained a wave equation and after employing the plane wave ansatz we get the dispersion relation for *Langmuir* waves

$$\boxed{\omega_e^2 = \omega_{pe}^2 + k^2 \gamma_e v_{th,e}^2}$$

Langmuir waves are oscillations of the electric field propagating to the plasma. Note that for $k = 0$ the dispersion relation for Langmuir waves recovers the plasma frequency. Because $k = 2\pi/\lambda$, plasma oscillations are plasma waves with a very large wavelength.

2 ION ACOUSTIC WAVES

So far we have only considered the high frequency motion of the electrons and ignored the contribution of the ions. However, at low frequencies the ion motion gets important. Let's have a look at the ratio of the electron plasma frequency ω_{pe} and the ion plasma frequency

$$\omega_{pi} = \left(\frac{n Z_i^2 e^2}{m_i \epsilon_0}\right)^{1/2}$$

for a plasma composed of electrons and protons. With $n_e = n_i$ and $Z_i = 1$ we find that

$$\frac{\omega_{pe}}{\omega_{pi}} = \left(\frac{m_i}{m_e}\right)^{1/2} \approx 43.$$

In this case the electrons can be treated to react on changes of the electric field with no inertia, i.e

$$e\delta E = -\gamma_e k_B T_e \frac{\partial \ln n_e}{\partial x}.$$

This means that there is a balance between the electron pressure and the electric force. We now rearrange the equation above for n_e

$$\ln n_e = \frac{1}{\gamma_e k_B T_e}\int e\delta E \, dx,$$

and replace δE by $\partial(\delta\Phi)/\partial x$

$$n_e = \exp\left\{\frac{e\delta\Phi}{\gamma_e k_B T_e}\right\}.$$

After linearizing the electron density

$$n_{e0} + \delta n_e \approx n_0 + \approx n_0 \frac{e\delta\Phi}{\gamma_e k_B T_e} + O(\delta\Phi^2)$$

and rearranging the expression we find the response of the electrons to low frequency ion oscillations

$$\frac{\delta n_e}{n_0} = \frac{e\delta\Phi}{\gamma_e k_B T_e}.$$

The continuity and momentum equations for the ion fluid are

$$\frac{\partial}{\partial t}\delta n_i = -n_i \frac{\partial}{\partial x}\delta v_i$$

$$\frac{\partial}{\partial t}\delta v_i = \frac{e}{m_i}\delta E - \nabla p_i.$$

Because the ions are cold we will drop the pressure term for now and we assume charge neutrality even in the perturbed densities, i.e. $\delta n_i = \delta n_e = \delta n$. After eliminating δE and δv_i from the set of equations we get

$$\frac{\partial^2}{\partial t^2}\delta n - \frac{\gamma_e k_B T_e}{m_i}\frac{\partial^2}{\partial x^2}\delta n = 0,$$

which is a wave equation. The corresponding dispersion relation

$$\boxed{\omega_{ia}^2 = \frac{\gamma_e k_B T_e}{m_i} k^2}$$

has the characteristics of a sound wave. For this reason low frequency ion waves are called *ion acoustics waves*.